that the author is writing with Andrews on the lost notebook [1].

We conclude with a few remarks about Ramanujan's methods. It has been suggested that he discovered his results by "intuition" or by making deductions from numerical calculations or by inspiration from Goddess Namagiri. Indeed, like most mathematicians, Ramanujan evidently made extensive calculations that provided guidance. However, Hardy and this writer firmly believe that Ramanujan created mathematics as any other mathematician would and that his thinking can be explained like that of other mathematicians. However, because Ramanujan did not leave us any proofs for the vast number of results found in his earlier notebooks and in his lost notebook, we often do not know Ramanujan's reasoning. As Ramanujan himself was aware, some of his arguments were not rigorous by then-contemporary standards. Nonetheless, despite his lack of rigor at times, Ramanujan doubtless thought and devised proofs as would any other mathematician.

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Jonathan M. Borwein

Ramanujan and Pi

Since Ramanujan's 1987 centennial, much new mathematics has been stimulated by uncanny formulas in Ramanujan's *Notebooks* (lost and found). In illustration, I mention the exposition by Moll and his colleagues [1] which illustrates various neat applications of Ramanujan's Master Theorem, which extrapolates the Taylor coefficients of a function, and relates them to methods of integration used in particle physics. I also note lovely work on the modular functions behind Apéry and Domb numbers by Chan and others [6], and finally I mention my own work with Crandall on Ramanujan's arithmetic-geometric continued fraction [12].

For reasons of space, I now discuss only work related directly to pi, and so continue a story started in [9], [11]. Truly novel series for $1/\pi$, based on elliptic integrals, were found by Ramanujan around 1910 [19], [5], [7], [21]. One is

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{1}$$

Each term of (1) adds eight correct digits. Though then unproven, Gosper used (1) for the computation of a then-record 17 million digits of π in 1985, thereby completing the first proof of (1) [7, Ch. 3]. Soon after, David and Gregory Chudnovsky found the following variant, which relies on the quadratic number field $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$, as is implicit in (1):

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{2}$$

Each term of (2) adds fourteen correct digits. (Were a larger imaginary quadratic field to exist with class number one, there would be an even more extravagant rational series for some surd divided by π [10].) The brothers used this formula several times, culminating in a 1994 calculation of π to over four billion decimal digits. Their remarkable story was told in a prize-winning *New Yorker* article [18]. Remarkably, (2) was used again in 2010 and 2011 for the current record computations of π to five and ten trillion decimal digits respectively.

Jonathan M. Borwein is Laureate Professor in the School of Mathematical and Physical Sciences at the University of Newcastle (NSW), Australia. His email address is `jon.borwein@gmail.com`.

Quartic Algorithm for π

The record for computation of π has gone from 29.37 *million* decimal digits in 1986 to ten *trillion* digits in 2011. Since the algorithm below, which found its inspiration in Ramanujan's 1914 paper, was used as part of computations both then and as late as 2009, it is interesting to compare the performance in each case: Set $a_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$; then iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}},$$

(3)

$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then a_k converges *quartically* to $1/\pi$; each iteration quadruples the number of correct digits. Twenty-one iterations produce an algebraic number that coincides with π to well over six trillion places.

This scheme and the 1976 Salamin–Brent scheme [7, Ch. 3] have been employed frequently over the past quarter century. Here is a highly abbreviated chronology (based on `http://en.wikipedia.org/wiki/Chronology_of_computation_of_pi`):

- 1986: David Bailey used (3) to compute 29.4 million digits of π. This required 28 hours on one CPU of the new Cray-2 at NASA Ames Research Center. Confirmation using the Salamin-Brent scheme took another 40 hours. This computation uncovered hardware and software errors on the Cray-2.
- January 2009: Takahashi used (3) to compute 1.649 trillion digits (nearly 60,000 times the 1986 computation), requiring 73.5 hours on 1,024 cores (and 6.348 Tbyte memory) of the Appro Xtreme-X3 system. Confirmation via the Salamin-Brent scheme took 64.2 hours and 6.732 Tbyte of main memory.
- April 2009: Takahashi computed 2.576 trillion digits.
- December 2009: Bellard computed nearly 2.7 trillion decimal digits (first in binary) using (2). This took 131 days, but he used only a single four-core workstation with lots of disk storage and even more human intelligence!
- August 2010: Kondo and Yee computed 5 trillion decimal digits, again using equation (2). This was done in binary, then converted to decimal. The binary digits were confirmed by computing 32 hexadecimal digits of π ending with position 4,152,410,118,610 using BBP-type formulas for π due to Bellard and Plouffe [7, Chapter 3]. Additional details are given at `http://www.numberworld.org/misc_runs/pi-5t/announce_en.html`. See also [4], in which analysis showing these digits appear to be "very normal" is made.

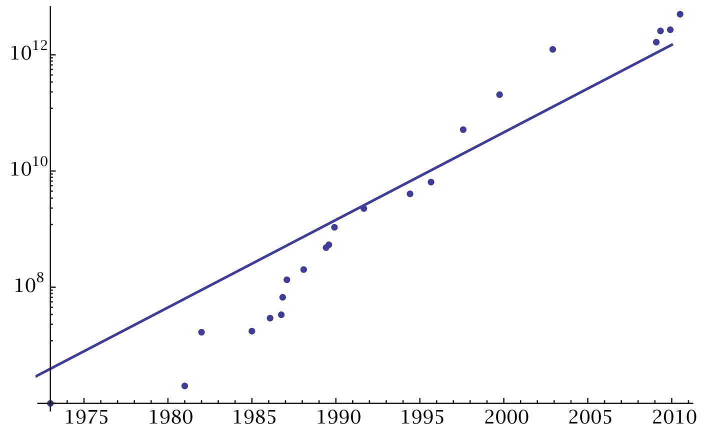

Figure 1. Plot of π calculations in digits (dots) compared with the long-term slope of Moore's Law (line).

Daniel Shanks, who in 1961 computed π to over 100,000 digits, once told Phil Davis that a billion-digit computation would be "forever impossible". But both Kanada and the Chudnovskys achieved that in 1989. Similarly, the intuitionists Brouwer and Heyting asserted the "impossibility" of ever knowing whether the sequence 0123456789 appears in the decimal expansion of π; yet it was found in 1997 by Kanada, beginning at position 17387594880. As late as 1989, Roger Penrose ventured, in the first edition of his book *The Emperor's New Mind*, that we likely will never know if a string of ten consecutive 7s occurs in the decimal expansion of π. This string was found in 1997 by Kanada, beginning at position 22869046249.

Figure 6 shows the progress of π calculations since 1970, superimposed with a line that charts the long-term trend of Moore's Law. It is worth noting that whereas progress in computing π exceeded Moore's Law in the 1990s, it has lagged a bit in the past decade. Most of this progress is still in mathematical debt to Ramanujan.

As noted, one billion decimal digits were first computed in 1989, and the ten (actually fifty) billion digit mark was first passed in 1997. Fifteen years later one can explore, in real time, multibillion step walks on the hex digits of π at `http://carmaweb.newcastle.edu.au/piwalk.shtml`, as drawn by Fran Aragon.

Formulas for $1/\pi^2$ and More

About ten years ago Jésus Guillera found various Ramanujan-like identities for $1/\pi^N$ using integer relation methods. The three most basic—and entirely rational—identities are

(4)
$$\frac{4}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13+180n+820n^2) \left(\frac{1}{32}\right)^{2n+1},$$

(5)
$$\frac{2}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1+8n+20n^2) \left(\frac{1}{2}\right)^{2n+1},$$

(6)
$$\frac{4}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1+14n+76n^2+168n^3) \left(\frac{1}{8}\right)^{2n+1},$$

where $r(n) := (1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2)/n!$.

Guillera proved (4) and (5) in tandem by very ingeniously using the Wilf-Zeilberger algorithm [20], [17] for formally proving hypergeometric-like identities [7], [15], [21]. No other proof is known. The third, (6), is almost certainly true. Guillera ascribes (6) to Gourevich, who found it using integer relation methods in 2001.

There are other sporadic and unexplained examples based on other symbols, most impressively a 2010 discovery by Cullen:

(7)
$$\frac{2^{11}}{\pi^4} \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^7 (\frac{3}{4})_n}{(1)_n^9}$$
$$\times (21+466n+4340n^2+20632n^3+43680n^4) \left(\frac{1}{2}\right)^{12n}.$$

We shall revisit this formula below.

Formulae for π^2

In 2008 Guillera [15] produced another lovely, if numerically inefficient, pair of third-millennium identities—discovered with integer relation methods and proved with creative telescoping—this time for π^2 rather than its reciprocal. They are based on:

$$(8) \quad \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} (6(n+x)+1) = 8x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2},$$

and

(9)
$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} (42(n+x)+5) = 32x \sum_{n=0}^{\infty} \frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}.$$

Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the *rising factorial*. Substituting $x = 1/2$ in (8) and (9), he obtained respectively the formulae

$$(10) \qquad \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (3n+2) = \frac{\pi^2}{4},$$
$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (21n+13) = 4\,\frac{\pi^2}{3}.$$

Calabi-Yau Equations and Supercongruences

Motivated by the theory of Calabi-Yau differential equations [2], Almkvist and Guillera have discovered many new identities. One of the most pleasing is

$$(11) \quad \frac{1}{\pi^2} \stackrel{?}{=} \frac{32}{3} \sum_{n=0}^{\infty} \frac{(6\,n)!}{(n!)^6} \frac{(532\,n^2+126\,n+9)}{10^{6n+3}}.$$

This is yet one more case where mysterious connections have been found between disparate parts of mathematics and Ramanujan's work [21], [13], [14].

As a final example, we mention the existence of *supercongruences* of the type described in [3], [16], [23]. These are based on the empirical observation that a Ramanujan series for $1/\pi^N$, if truncated after $p-1$ terms for a prime p, seems always to produce congruences to a higher power of p. The formulas below are taken from [22]:

(12)
$$\sum_{n=0}^{p-1} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^3 (\frac{3}{4})_n}{2^{4n}\,(1)_n^5} \left(3+34n+120n^2\right) \equiv 3p^2 \pmod{p^5},$$

(13)
$$\sum_{n=0}^{p-1} \frac{(\frac{1}{4})_n (\frac{1}{2})_n^7 (\frac{3}{4})_n}{2^{12n}\,(1)_n^9} \left(21+466n+4340n^2+20632n^3+43680n^4\right)$$
$$\stackrel{?}{\equiv} 21p^4 \pmod{p^9}.$$

We note that (13) is the supercongruence corresponding to (6), while for (12) the corresponding infinite series sums to $32/\pi^4$. We conclude by reminding the reader that all identities marked with '$\stackrel{?}{=}$' are assuredly true but remain to be proved. Ramanujan might well be pleased.

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The second installment of this article—with pieces by Ken Ono, K. Soundararajan, R. C. Vaughan, and S. Ole Warnaar—will appear in the January 2013 issue of the Notices.